Muon reconstruction with AMANDA

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Abstract
This document describes the method of muon track reconstruction in AMANDA.
1 Introduction

The main aim of the AMANDA detector is to detect the Čerenkov light from charged relativistic particles such as muons. Clearly identified up-going muon tracks are a key signature for charged current interactions of high energy muon neutrinos, which are expected to be emitted from astrophysical accelerators.

![Cherenkov Wave Front](image)

As sketched in figure 1 and 2, muon tracks emit Čerenkov photons into a cone of the angle \( \theta_c \approx 41^\circ \)

\[
\cos \theta_c = \frac{1}{n \beta} = \frac{c_{\text{ice}}}{v_\mu}.
\]  

(1)

\( n \) is the index of refraction, \( \beta \) the rapidity of the muon. For relativistic tracks \( (v_\mu \approx c_{\text{vac}}) \) the Čerenkov angle is fixed \( \theta_c \approx 41^\circ \). During the propagation through the ice, photons may disappear due to absorption or change their direction due to scattering. The muon track can be reconstructed by the measurement of the photon arrival times in several PMTs.

2 Reconstruction method

2.1 Arrival time of photons

The geometrical arrival time \( t_{\text{geo}} \) at a PMT is given by (see figure 2)

\[
t_{\text{geo}} = t_0 + \frac{\vec{p} \cdot (O\hat{M} - \vec{r}_0) + d \cdot \tan \theta_c}{c_{\text{vac}}}.
\]  

(2)

It is convenient to define a time residual \( t_{\text{res}} \) for a given track and experimentally measured...
Figure 3: Time residuals as predicted from a Monte Carlo Simulation of isotropic minimum ionizing muons. The figure integrates over all distances of muons to the optical modules. The figure contains a realistic PMT simulation; hits with an amplitude $\geq 1.5$ p.e. are taken as 2 p.e. signals. No noise hits have been simulated.

Times $t_{\text{exp}}$ via

$$t_{\text{res}} = t_{\text{exp}} - t_{\text{geo}}.$$  

(3)

Negative values of $t_{\text{res}}$ would correspond to a causality violation and are thus only allowed in case of uncorrelated noise hits. Experimentally measured $t_{\text{res}}$ can be written as

$$t_{\text{res}} \approx \pm \sigma_{\text{PMT}} + t_{\text{sec}} + t_{\text{scatt}}.$$  

(4)

The accuracy of the measured time residuals is usually limited by the time-jitter of the photomultiplier $\sigma_{\text{PMT}}$ (and the accuracy of its calibration). Additional time delays may occur due to secondary showers along the muon track $t_{\text{sec}}$ and due to scattering of the photon, which leads to a longer path length and therefore an additional time delay $t_{\text{scatt}}$.

Figure 3 shows time residuals as obtained from simulation of isotropic muons in AMANDA-B. A clear peak of almost unscattered photons is visible at $t_{\text{res}} \approx 0$. These hits are in the following called direct hits (see also section 3.3). Scattered photons are visible in a long tail towards larger time residuals. In case of larger signal amplitudes (2p.e.) muons are typically closer to the photomultiplier. Therefore the number of scattered photons and thus delayed hits is smaller. Secondary showers along the muon track emit their light with a broader angular distribution than the muon track and are thus visible under different angles than the Čerenkov angle. Especially bright bremsstrahlung events may lead to delayed hits with $t_{\text{res}} \gg 0$ but not to earlier hits ($t_{\text{sec}} \geq 0$). However most photons from secondary processes arrive at the same time as the light from the muon itself (e.g figures 7.60 to 7.63 in [11]).

2.2 Reconstruction model

The simplest method of reconstruction can be achieved by the variation of the track parameters $\vec{r}_0, t_0, \vec{p}, E$ from initial values (first guess) until the spectrum of observed time residuals fits best
to the expectation. In case of no scattering this suggests to minimize

$$\chi^2 = \sum_{i=1}^{n_{\text{hits}}} \frac{t_{\text{res}}^2}{\sigma_{\text{PMT}}^2}.$$  

(5)

For AMANDA, a large fraction of photons arrive delayed. The time delay due to scattering is non-gaussian. It also depends strongly on the distance to the muon track. Therefore a simple $\chi^2$-method like eq.5 has not been successful.

![Diagram of reconstructed muon event in AMANDA-B-4 detector](image)

Figure 4: Example of a reconstructed muon event in the AMANDA-B-4 detector (Monte Carlo). The picture shows the true MC muon track ($\mu$-generated) and two different attempted reconstructions. Lines connect each hit PMT (dots) to each track under the Čerenkov angle. For direct hits, arriving at the expected time $t_{\text{res}} = 0$, the line ends at the PMT. Lines, which overshoot the PMT correspond to delayed photons. Shorter lines indicate hits, which occur too early and would thus violate causality.

Instead one does a maximum likelihood reconstruction by minimizing $-\log(\mathcal{L})$, where $\mathcal{L}$ is the total likelihood of all arrival times in the event. It is given by

$$\mathcal{L} = \prod_{i=1}^{N_{\text{hit}}} P(t_{\text{res}}^i (d_i, \eta_i, a_i, ...)).$$  

(6)

$P$ the normalized probability to observe the time residual $t_{\text{res}}^i$. It depends on several parameters, such as the distance of the PMT to the muon $d$, it's orientation towards the Čerenkov cone $\eta$ or
the measured amplitude $a_i$. In the AMANDA case, where scattering is a strong effect the time residual is approximately given by

$$P(t_{res}) \approx P(t_{scatt}) .$$

Figure 4 demonstrates the comparison of a traditional $\chi^2$ reconstruction based on a pure Čerenkov model ($t_{res} \approx 0$) and a likelihood reconstruction based on a model, which takes into account the photon scattering. While the $\chi^2$ fit tries to average out direct hits with severely delayed hits, the Likelihood fit actually expects these large delays for large distances and reconstructs the track well; in this example better than 1°.

The main challenge in terms of likelihood minimization is to develop a normalized probability function. The additional time delay due to scattering depends on multiple parameters, such as the distance to the track and the optical parameters of the ice. This has been investigated by a detailed photon propagation Monte Carlo calculation [4]. The resulting photon density and time delay distributions are archived in large multi-dimensional look-up tables and used for the detector simulation [4, 5].

The general approach in terms of reconstruction is to parameterize the contents of these tables using a simple function. Two parameterizations, based on different functions have been developed independently to describe the probability distributions of a certain time-delay due to scattering. The in the following described Pandel-function is essentially a Gamma distribution with an additional exponential absorption term. It thus fulfills the requirements of simplicity and speed and normalization. It is also very handy in terms of calculation, e.g. integration, which allows the development of a multi-PE or a Poisson-saturated distribution. The other approach uses an F-function, which is connected to an exponential tail [2]. Both parameterizations yield about the same reconstruction accuracy. It is also under discussion to use the look-up tables themselves in a future reconstruction program.

2.3 The Pandel function

The Pandel function is motivated by the scattering problem of a point-like monochromatic isotropic light source and a point-like receiver. The probability distribution $p(d, t)$ of time delays $t_{scatt}$ can be written according to [7]

$$p(t, d) = \frac{1}{N} \cdot \frac{\tau c_{ice}^{d/\lambda}}{\Gamma(d/\lambda)} \cdot d^{d/\lambda - 1} \cdot e^{-(t + c_{ice} t / X_0 + d / X_0)} .$$

without special assumptions on the actual optical parameters. $c_{ice} = c_{vac} / n$ is the speed of light in ice, $X_0$ the absorption length and $N$ the normalization. The free parameters $\tau(d, \ldots), \lambda(d, \ldots)$ are functions of the distance $d$ and the other free parameters such as the orientation of the PMT. In [7] it is shown, that they can be written as a series expansion of the scattering length $\lambda_s$,

$$\tau(d) = \sum_i \tau_i \cdot \lambda_i^{\gamma_i} \cdot d^{1 - \gamma_i} \quad \lambda(d) = \sum_i \lambda_i \cdot \lambda_i^{\gamma_i} \cdot d^{1 - \gamma_i}$$

with arbitrary values of $\gamma_i$, $\tau_i$ and $\lambda_i$. Specific values of e.g. $\lambda_s$ or $\cos \theta_s >$ result in different functions for $\tau(d)$ and $\lambda(d)$.

The function can be integrated

$$P(t, d) = \int_0^t p(t, d) dt = \frac{e^{-d/X_0} \cdot (1 + \frac{\tau \cdot c_{ice}}{X_0})^{-d/\lambda}}{N \Gamma(d/\lambda, t \cdot (1 / \tau + c_{ice} / X_0))} .$$

5
log10 ( p(t)/ns )

Figure 5: 3-d plot of the Pandel distribution (not normalized) as a function of the time delay $t_{\text{scatt}}$ and distance $d$.

with the incomplete Gamma Function

$$
\Gamma_{I}(y, x) \equiv \frac{1}{\Gamma(y)} \cdot \int_{0}^{x} e^{-t} \cdot t^{y-1}dt .
$$

(11)

The normalization is thus given by

$$
N = \int_{0}^{\infty} p(t, d)dt = e^{-d/X_0} \cdot \left( 1 + \frac{\tau \cdot c_{\text{ice}}}{X_0} \right)^{-d/\lambda} ,
$$

(12)

and the integral in eq.10 by the $\Gamma$ term only. The function is already normalized ($N = 1$) in case of an infinite absorption length. The first derivative is

$$
\dot{p}(t, d) \equiv \frac{dp}{dt} = p(t, d) \cdot \left( \frac{d/\lambda - 1}{t} - \frac{1}{\tau} - \frac{c_{\text{ice}}}{X_0} \right) .
$$

(13)

Figure 5 shows a 2 dimensional plot of the distribution eq.8. For small distances the function has a pole at $t = 0$. Going to larger values of $d$, larger delay times become more likely. For distances larger than the critical value $d = \lambda$ the function suddenly changes and the probability of undelayed photons becomes zero.

Our strategy to parameterize the delay probabilities can be outlined as follows: This simple function eq.8 is fit to the distributions of delay times. These have been calculated by a detailed photon propagation Monte Carlo for the Čerenkov light from muons at various distances [4]. Free parameters are $\tau$, $\lambda$, $X_0$ and the effective distance $d_{\text{eff}}$. These parameters are parameterized as a function of the true distance to the muon $d_\mu$ and the orientation of the PMT towards the Čerenkov cone $\eta$.

Already for a very simple ansatz (see also eq.9), taking $\tau$, $\lambda$, $X_0$ constant with the distance and the orientation, the optics in AMANDA is described reasonably well as shown in figure 6. By
an angle dependent linear dependence of the effective distance also different PMT orientations are reasonably described. The following parameters are currently most favored:

\[
\begin{align*}
\tau &= \tau_1 = 557\,ns \\
\lambda &= \lambda_1 = 33.3\,m \\
X_0 &= 98\,m \\
d_{\text{eff}} &= d_0 + d_1 \cdot d_\mu \\
d_1 &= 0.84 \\
d_0 &= 3.1m - 3.9m \cdot \cos \eta + 4.6 \cdot \cos^2 \eta
\end{align*}
\] (14)

However the price for this simple overall description is a limited accuracy, especially for intermediate distances \(d_\mu \approx 30m - 50\,m\), where \(d \approx \lambda\).

The reason to use this simple function (designed for point sources) in order to parameterize Čerenkov light from muon tracks is best motivated by the following statement: “I think the Pandel function fits the data well, because it has some physics in it, and because it has many parameters to make it flexible for the physics that is not understood” [3].

### 2.4 Patched Pandel

Though we achieved with eq.8 a simple normalized likelihood description, it is not yet suitable for our purposes. Eq.8 is undefined for negative times and does not contain any PMT jitter. For short distances it has an infinite pole at most probable time 0.0\(ns\). Thus the used numerical minimization algorithms have difficulties to converge correctly. A best solution could be to convolute eq.8 with a Gaussian PMT jitter. Unfortunately this integral has not been solved yet. Instead a simple modification to the original function eq.8, called “Patched” Pandel function \(\hat{p}(t, d)\), was developed.

Basic idea is that the PMT-jitter is small compared to time-scales of scattered photons; it therefore plays only a role for direct photons \(t_{\text{scatt}} \approx 0 < t_1\), with times smaller than \(t_1\) of the
Figure 7: Comparison of the Pandel function and the “patched” function for a distance of 20m.

order of a few 10 ns. We use therefore following patched function:

\[
\hat{p}(t, d) = \begin{cases} 
G(t, d) = \frac{N_g(d)}{\sqrt{2\pi} \cdot \sigma_g} \cdot \exp \left( -\frac{t^2}{2\sigma_g^2} \right) & \text{for } t < 0 \\
L(t, d) = a_0(d) + a_1(d) \cdot t + a_2(d) \cdot t^2 + a_3(d) \cdot t^3 & \text{for } 0 < t < t_1 \\
p(t, d) & \text{for } t > t_1
\end{cases}
\]  

for \( t > t_1 \).

(15)

The result can be seen in figure 7. \( p(t) \) is extended to \(-\infty\) by half a Gaussian, which peak is connected to \( p(t, d) \) by a 3rd order degree polynomial, in order to achieve a function which is also continuous in its first derivative. The normalization condition has to assure that \( \hat{p} \) is still normalized. This problem can be solved using the following conditions:

\[
\begin{align*}
\sigma_g & \approx \sigma_{PMT} \\
t_1 & = \sqrt{2\pi} \cdot \sigma_g \\
G(0, d) & = L(0, d) \\
L(t_1, d) & = p(t_1, d) \\
\dot{G}(0, d) & = \dot{L}(0, d) \\
\dot{L}(t_1, d) & = \hat{p}(t_1, d) \\
\int_0^{t_1} p(t, d) \, dt & = \int_{-\infty}^{0} G(t, d) \, dt + \int_{t_1}^{t_1} L(t, d) \, dt
\end{align*}
\]

(16)

The solution is

\[
\begin{align*}
N_g & = a_0 \cdot t_1 \\
a_0 & = \frac{P(t_1, d)}{t_1} - \frac{p(t_1, d)}{2} + \frac{\hat{p}(t_1, d) \cdot t_1}{12} \\
a_1 & = 0
\end{align*}
\]

(17)
\[
a_2 = -\frac{3P(t, d)}{t_1^3} + \frac{9p(t_1, d)}{2t_1^2} - \frac{5\dot{p}(t_1, d)}{4t_1}
\]
\[
a_3 = -\frac{2P(t, d)}{t_1^3} - \frac{3p(t_1, d)}{t_1^3} + \frac{7\dot{p}(t_1, d)}{6t_1^2}
\]
with \(P(t, d)\) and \(\dot{p}(t, d)\) defined in eq.10 and 13. The only remaining free parameter is \(\sigma_g\). Good reconstruction results are achieved for \(10\text{ns} \leq \sigma_g \leq 20\text{ns}\).

### 2.5 Noise hits

The problem of (white) noise hits, which occur uncorrelated with the muon, is equivalent to a probability density \(p_0\) constant\(^1\) in time. This is already solved by introducing a Maximum value to eq.6

\[
-L_i = \text{MIN}(-\log(\dot{p}(t, d)), -\log(p_0))
\]  \hspace{1cm} (18)

More correctly one has to add \(p_0\) to \(\dot{p}(t, d)\) and renormalize afterwards. In practice this equivalent to an additional constant bias in eq.6 (∼ \(N_{\text{hits}}\cdot \log(p_0)\)), which thus only slightly affects the minimization process. For a noise rate of 1kHz one gets \(p_0 \approx 1 \cdot 10^{-6}\text{ns}^{-1}\), which is further reduced in case of noise hit suppression (section 3.2).

### 2.6 Multi photon and Poisson saturated likelihood

The previous calculations only described the probability density for individual photons. In case of multiple photon hits in one PMT it is often desirable to have a probability density for the first of these photons. E.g. in the case of AMANDA channels with electrical cables can only resolve photons, which are separated by more than a few hundred ns. Since all photons travel independently the probability distribution for the time of the first photon \(p_1(t, d)\) of \(N\) photons can be calculated from the probability distribution of a single photon \(p(t, d)\) via

\[
p_1(t, d) = N \cdot p(t, d) \cdot \left(\int_t^\infty p(t, d)dt\right)^{(N-1)} = N \cdot p(t, d) \cdot (1 - P(t, d))^{(N-1)}.
\]  \hspace{1cm} (19)

The distribution for the \(k\)th photon may be written as

\[
p_k(t, d) = N \cdot p(t, d) \cdot (1 - P(t, d))^{(N-k)} \cdot (P(t, d))^{(k-1)}.
\]  \hspace{1cm} (20)

The single photon distribution \(p\) and its cumulative \(P\) are given in eq.8 and eq.10. With the differential

\[
\dot{p}_1(t, d) = N \cdot \dot{p}(t, d) \cdot (1 - P(t, d))^{(N-1)} - N \cdot (N-1) \cdot p^2(t, d) \cdot (1 - P(t, d))^{(N-2)}
\]  \hspace{1cm} (21)

and the cumulative

\[
P_1(t, d) = \int_0^t p_1(t, d)dt = 1 - (1 - P(t, d))^N
\]  \hspace{1cm} (22)

the same patch algorithm may be used as for eq.8 (see section 2.4). With \(P(\infty, d) = 1\) and therefore \(P_1(\infty, d) = 1\) one sees, that eq.19 is already normalized. In many cases, where the amplitude is not measured precisely enough it is desirable to follow a different approach [2, 3, 8]. Instead of measuring one calculates the expected amplitude \(\mu\) as a function of the distance \(d\) (e.g. via eq.29). Using this mean amplitude \(\mu\) one may calculate the

\(^1\)Of course the hit cleaning procedure (section 3.2) may change this.
probability distribution of the first photon. Since the actual number of photons is unknown, one has to convolute the multi PE distribution $p_N^1(t, d)$ with the Poisson probability of the mean $\mu$ to get $N$ photons

$$p_\mu(t, d) = \sum_{i=1}^{\infty} \frac{\mu^i e^{-\mu}}{i!} \cdot p_i^1(t, d).$$

(23)

This so called poisson saturated distribution was calculated by [2, 3]

$$p_\mu(t, d) = \frac{\mu}{1 - e^{-\mu}} \cdot p(t, d) \cdot e^{-\mu P(t, d)}.$$ 

(24)

Again same patch algorithm of section 2.4 may be applied using the differential

$$\dot{p}_\mu(t, d) = p_\mu(t, d) \cdot \left( \frac{\dot{p}(t, d)}{p(t, d)} - \mu \cdot p(t, d) \right)$$

(25)

and the cumulative

$$P_\mu(t, d) = \int_0^t p_\mu(t, d) dt = \frac{1 - e^{-\mu P(t, d)}}{1 - e^{-\mu}}.$$ 

(26)

2.7 Hit probability and energy reconstruction

![Graphs](A)(B)(C)

Figure 8: Hit probability for different values of the $\epsilon$ parameter. (A) $P_{hit}(d)$ for $\epsilon = 0.5, 1, 2, 4, 10$ (from left to right)  (B) $P_{hit}(d)$, $P_{nohit}(d)$, $P_{hit}(d) \cdot P_{nohit}(d)$, for $\epsilon = 2$  (C) $P_{hit}(d) \cdot P_{nohit}(d)$ for $\epsilon = 0.5, 1, 2, 4, 10$ (from left to right)

Eq.(8) already includes an absorption term for photons. When thinking about normalizing eq.8 to the hit probability instead of 1 one may interpret eq.12 as the hit-probability of a PMT, of a certain, but unspecified sensitivity and a certain amount of injected photons

$$P^0_{hit}(d) \equiv N(d).$$

(27)

Assuming that the expected number of photoelectrons $\mu$ is proportional to the relative sensitivity of the PMT $\epsilon_{PMT}$ and the intensity of the muon light $\epsilon_E$ relative to the values in eq.27 one can write

$$\mu \propto \epsilon = \epsilon_{PMT}(\eta) \cdot \epsilon_E.$$ 

(28)
Using Poisson statistics one finds the identity
\[\mu = - \log(P_{\text{nohit}}) = \epsilon \cdot \log\left(P_{\text{nohit}}^0(d)\right) = \epsilon \cdot \log\left(1 - P_{\text{hit}}^0(d)\right).\] (29)

Therefore the hit-probability of an AMANDA-PMT to a muon of certain energy can be written as
\[P_{\text{nohit}}(d) = 1 - P_{\text{hit}}(d) = (1 - P_{\text{hit}}^0(d))^\epsilon,\]
\[P_{\text{hit}}(d) = 1 - P_{\text{nohit}}(d) = 1 - (1 - P_{\text{hit}}^0(d))^\epsilon.\] (30)

Figure 8 (A) shows the hit probability for different values of $\epsilon$. A typical value for a head-on illumination of an AMANDA PMT by a minimum ionizing muon would correspond to about $\epsilon \approx 2.7$. The mean visual range of a muon corresponds to about the distance, where $P_{\text{hit}}(d) = P_{\text{nohit}}$; the Maximum of $P_{\text{hit}}(d) \cdot P_{\text{nohit}}(d)$. This is shown in (B) together with $P_{\text{hit}}(d)$, $P_{\text{nohit}}$ for $\epsilon = 2$ and for different values of $\epsilon$ in (C). Note that $\epsilon$ is about proportional to the muon energy.

The absolute value of $\epsilon$ is unknown and has to be calibrated, e.g. by Monte Carlo. A preliminary, experimentally not yet verified parameterization has been fit to data generated by a detector MC, which includes a realistic PMT simulation. One finds good agreement for
\[
\begin{align*}
\tau &= 550\text{ns} \quad \lambda = 22\text{m} \quad X_0 = 98\text{m}, \\
\epsilon_{\text{PMT}} &\approx 1 + 0.35 \cdot \cos(\eta), \\
\epsilon_{\text{E}} &\approx 2 \cdot (1 + 9.4 \cdot 10^{-4}\text{GeV}^{-1} \cdot E_\mu),
\end{align*}
\] (31)

with the $\eta$ the relative angular orientation of the PMT.

A realistic description also requires to include the probability of a noise hit $P_{\text{noise}}$ via
\[P_{\text{hit, tot}} = P_{\text{hit, \mu}} + P_{\text{noise}} - P_{\text{hit, \mu}} \cdot P_{\text{noise}}\] (32)

A likelihood function for the hit probability is easily constructed
\[\mathcal{L} = \prod_{\text{all hit PMT}} P_{\text{hit}}(d, \ldots) \times \prod_{\text{not hit PMTs}} P_{\text{nohit}}(d, \ldots).\] (33)

This function can be used not only as a second model for track reconstruction, but also for the reconstruction of the muon energy, when leaving $\epsilon$ as a free fit parameter.

In a Monte Carlo calculation for the AMANDA-II detector it was found that this simple method, which uses only the hit probability information, already an energy resolution of $\sigma(\Delta \log_{10} E) \approx 0.3$ can be achieved. It is obvious that a future inclusion of the actually measured pulse-height information and the energy dependent time delay distribution like e.g. eq.24, should further improve these results. Therefore these results are preliminary. However the unexpected good results already for this method can be motivated by figure 8 (C). The above likelihood reconstruction essentially reconstructs the mean visual range of the muon. For different values of the muon energy $\epsilon \propto E_\mu$ the mean visual range is seen to be clearly separated. Problematic in the energy reconstruction is the strong stochastic character of the muon energy loss. Since this method is sensitive to larger distance scales, where photons are starting to become isotropic, fluctuations are averaged out. This is an interesting example where scattering improves the experimental result!

The above method is already used in the current analysis scheme as a quality criteria to identify badly reconstructed events (see section 3.3). When using only the hit times, a poorly reconstructed track may pass close to not hit PMTs or far distant from hit PMTs. This consequently corresponds to a poor value of the $P_{\text{hit}}$ likelihood of eq.33.
3  Signal processing

The actual processing of the data is carried out in several steps as sketched in the following diagram

\[
\text{Data} \Rightarrow \text{1st guess} \Rightarrow \text{Likelihood} \Rightarrow \text{Q-analysis}
\]

After the experimental data has been preprocessed, e.g. calibrated, a 0-th approximation, also called 1st guess, is calculated to get an initial approximation of the muon track. The following track fit uses these results as initial starting values. The track fit may be followed by additional fits using different algorithms or e.g. an energy reconstruction. As final step various criteria are applied to select events of certain interest or to reject poorly reconstructed events. These criteria may be also applied as filters in between the different steps, e.g. after the 1st guess, in order to reduce the amount of data and speed up the processing. The effect of these criteria as well as the quality of the reconstructions can be strongly improved by suppression of certain hits, e.g. due to noise. This process is called hit-cleaning. One has to keep in mind that the effective area, angular acceptance and resolution, depend depend strongly on the above applied procedure.

3.1 1st guess: Line-fit

First guess methods are analytical methods to calculate an initial approximation of the muon track. A large variety of them are known, such as a plane wave fit or the tensor of inertia of hit PMT positions. The presently most successful algorithm is the so-called line fit [10, 6].

Starting point is the minimization of a functional \( \chi^2 = \sum_i (t_{res})^2 \) similar to eq.5. A simple analytic solution to this problem can be achieved when using the approximation

\[
\vec{r}_i' \approx \vec{r}_0' + \vec{v}(t_i - t_0)
\]

where \( \vec{r}_i' \) is the location of the PMT belonging to the i-th hit (time \( t_i \)) and \( \vec{r}_0' \) the coordinate of the muon at \( t_0 \) and \( \vec{v} \) the speed vector of the muon. In order to improve the angular resolution, each hit can be weighted with its amplitude \( a_i \) to the power \( \gamma \). Calculating \( t_0 \equiv 0 \)

\[
\chi^2 = \sum_i a_i^2 \cdot (\vec{r}_i' - \vec{r}_0' - \vec{v} \cdot t_i)^2, \quad \frac{d\chi^2}{d\vec{r}} = 0, \quad \frac{d\chi^2}{d\vec{v}} = 0
\]

(35)

gives the analytical solution

\[
\vec{v} = \frac{< \vec{r}_i t_i > - < \vec{r}_i > < t_i >}{< t_i^2 > - < t_i >^2}, \quad \vec{r} = < \vec{r}_i > - \vec{v} < t_i >
\]

(36)

\(< ... > \) indicates the calculation of the amplitude weighted mean values. The resulting speed \( \vec{v} \) gives the direction of the track. It’s absolute value can be smaller than the speed of light \( v_{\text{vac}} \), due to the above approximation and the scattering delay of photons. It’s absolute value is a good quality criteria, e.g. \([6]\). Larger values indicate a topology of times, which line up more nicely than for smaller values. It is important to note, that the results of this analytical method are rather sensitive to the applied hit-cleaning algorithms(see section 3.2).

After the calculation of this initial track approximation the convergence of the following reconstruction can be further improved: One calculates all time residuals for the above track, this time according to the full Čerenkov model (see section 2.1). Now the above time \( t_0 \) is shifted in such a way that e.g. the earliest time residual corresponds to \( t_{res} = 0 \).
3.2 Hit cleaning

The quality of the reconstruction, especially the line-fit, can be significantly improved, if certain hits are not used, but rather a sub-sample, which are regarded as “good” hits. The applied algorithms to select these hits differ for each stage of analysis or reconstruction, depending on the actual requirements. Typical examples of hits to be suppressed are noise- or severely delayed hits. Following are examples of typical criteria, which can be applied to remove hits.

- Hits in PMTs with non-stable count rates or other strange features.
- Hits with very small amplitudes or pulse durations. These are likely to be due to cross-talk in the electrical cable.
- Hits with times outside a certain window (~ 5μs) around the time of the trigger.
- Coincident hits (e.g. Δt ≤ 500ns) in close-by PMTs (e.g. less than ~ 50m). A similar criteria are coincidences along one string.
- Accept only the first hit in each PMT.

3.3 Quality analysis

A series of good quality criteria have been developed to select events which are “well” reconstructed. “Well” usually means a compromise, which optimizes between the number of passing signal events (corresponding to a larger sensitivity) and the number of remaining background events (e.g. with a poor angular resolution).

Some often used criteria are summarized in table 1. One distinguishes between topological and reconstruction derived parameters. The first type are usually easily (fast) calculated and do not depend on detailed model assumptions. Examples are COGZ but also θLF and |v|LF. They are often used for pre-filtering of events, also online at the South-pole. The second class are based on information of the reconstruction, e.g. the Likelihood or error matrix, or are calculated properties, based on the fitted track parameters, e.g. the number of direct hits.

The application of a quality criteria is demonstrated in figure 9, which shows the zenith angle distribution of reconstructed Monte Carlo events in AMANDA-B. One sees, that after requiring a harder cut from the top to the bottom, the reconstructed distribution fits better to the simulated distribution. While the top figures show a certain fraction of events, which have been reconstructed even into the wrong hemisphere, these events are strongly suppressed at the bottom. However the angular acceptance of the detector changes. While without cuts (top left) the angular acceptance is relatively flat, after application (bottom left) more vertical tracks are likely to pass the Ndirect criteria.

4 Results from AMANDA-B4

4.1 Reconstructed atmospheric muons

The most important test of the reconstruction is the comparison between data and Monte Carlo for down-going muons. Usually the distributions of characteristic variables (see table 1) are compared for various levels of cuts. A detailed discussion goes beyond the scope of this document and is discussed elsewhere (e.g.[1, 2, 5, 9]). The angular resolution depends on the level of cuts
| \( \theta_{rec} \) | Reconstructed zenith angle |
| \( N_{direct} \) | Number of direct hits. This are the number of hits with a time residual \( t_{res} \) within the time of \(-15\)ns and \(+15\)ns (A), \(+25\)ns (B), \(+75\)ns (C) |
| \( Z_{direct} \) | Projected length of direct hits. The coordinates of all PMT, which measured a hit with a small time residual, are projected on the reconstructed track. The length of this projection can be interpreted as fit lever arm in terms of unscattered photons |
| \(-L_{time}/N_{ch} \) | Likelihood per number of hit channels of the track reconstruction using the hit times. Analogous to a \( \chi^2 \)-cut smaller values indicate a better fit |
| \(-L_{P_{hit}}/N_{led} \) | Likelihood of the energy reconstruction per number of working channels (see section 2.7). This corresponds to an analysis of the hit probability of PMTs with a hit and the no-hit probability of channels without a hit. Smaller values correspond to a higher (better) overall probability |
| \( COG_z \) | Center of Gravity of z-positions of hit channels. Bad reconstructed events often tend to originate from bright showers outside, especially from below the detector. Requiring the centre of hits sufficiently inside the detector, rejects these events |
| \( \theta_{LF} \) | Zenith angle of the Line-fit (see section 3.1). This parameter is especially usefull for event pre-filtering |
| \( |\vec{v}|_{LF} \) | Speed of the Line-fit. This a powerful parameter which depends on the topology of hit times in the event. Values close to the speed of light indicate a "good" — values smaller than e.g. \( 0.1 \)m/ns a "difficult" time pattern |

Table 1: Summary of quality criteria.

which have been applied. Typical values for the median angular mismatch angle are AMANDA-B4: 3.5°, AMANDA-B10: 2.4° and about 1.2° for AMANDA-II.

4.2 AMANDA — Spase-2 coincidences

An important test of the reconstruction can be done by analyzing events, which are coincident with the air-shower detector Spase-2. Spase-2 is located at the ice surface at a distance of a 370m from AMANDA. Coincident events correspond to a few parallel muons, which arrive under an almost vertical zenith angle of 10° to 14° in AMANDA. The lateral spread of muons around the shower core is typically a few meters. The shower core is reconstructed by Spase-2 within a few meters and 1° to 2° degrees resolution, depending on the shower size. The accuracy of the reconstruction can be tested by comparing the AMANDA reconstruction to the extrapolated shower core at the depth of AMANDA.

Coincident events, which point towards AMANDA, are reconstructed without using the available Spase-2 information. The difference of the reconstructed zenith angle is shown in figure 10 (A). It can be seen that a large fraction of events pass the application of relatively weak cuts. The difference of zenith angles of about 3.5° (inclusive the Spase reconstruction error). No shift is visible. This shows that AMANDA can exactly point towards the direction of Spase-2. When comparing the depth of tracks (when passing closest to the centre of the detector) one sees no
Figure 9: Zenith angle distribution of isotropic simulated up-going muons (left) and atmospheric down-going muons (right). The figures show the true (MC) direction (solid) and the reconstructed (MC) direction (dotted). No cuts are applied to the top pictures, the middle (bottom) show the distribution of events, which pass the criteria of having more than 3 (4) direct hits.

vertical shift in figure 10 (B) and a spread of a few m around the extrapolated depth. Since this distribution contains the spread of muons around the shower core it can be concluded that also the spatial resolution of the Amanda reconstruction is sufficiently good. The error in the absolute depth of the AMANDA detector seems to be less than a few m.

5 Conclusion and outlook

The reconstruction of muon events can be successfully done by a Likelihood analysis of the times of hit PMTs, using a simple parameterization of time delays due to scattering. Results of the early AMANDA-B-4 (1996) detector are encouraging. However the method is still evolving, presently focusing on the analysis of the data collected with the full AMANDA-B-10 detector (1997). Especially in the development of appropriate quality criteria to select good events and reject poorly reconstructed events significant improvements can be expected.
Figure 10: Reconstruction of Spase-2 Amanda-B-4 coincidences. (A) Zenith difference of the Spase and Amanda reconstruction ($\theta_{\text{AMANDA}} - \theta_{\text{Spase}}$) The shaded area correspond to the events after application of quality criteria ($N_{\text{direct,75ns}} \geq 5$, $Z_{\text{direct,25ns}} \geq 50$ m, $-L_{\text{time}}/N_{\text{ch}} \leq 12$). The Gauss fit gives a mean $-0.13^\circ \pm 0.18^\circ$ and a sigma of $3.65^\circ \pm 0.17^\circ$. (B) Difference of the z-component of the shower core position and the reconstructed track (impact parameter) ($Z_{\text{AMANDA}} - Z_{\text{Spase}}$). The mean of the distribution is $+0.25$ m.

The reconstruction method itself is still being improved. One idea is to use the tabulated photon calculations [4] instead of the here presented parameterizations. A reconstruction of point-like shower events is under development. In terms of energy reconstruction further improvements are obvious by using the energy dependent PMT amplitudes and time distributions.

References